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RESTRICTIONS ON SMALLEST COUNTEREXAMPLES TO THE 5-FLOW CONJECTURE

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Using counting arguments, we show that every smallest counterexample to Tutte's 5-flow Conjecture (that every bridgeless graph has a nowhere-zero 5-flow) has girth at least 9.

1. Introduction

A graph admits a *nowhere-zero k-flow* if its edges can be oriented and assigned numbers $\pm 1,\dots,\pm(k-1)$ so that for every vertex, the sum of the values on incoming edges equals the sum on the outgoing ones. It is well-known that a graph with a bridge (1-edge-cut) does not have a nowhere-zero k-flow for any $k \geq 2$ (see, e. g., [3,9,15]). The famous 5-*flow conjecture* of Tutte [13] is that every bridgeless graph has a nowhere-zero 5-flow.

Let G be a counterexample to the 5-flow conjecture of the smallest possible order. It is well-known (see cf. Jaeger [3] and Zhang [15]) that G must be a *snark* which is a cyclically 4-edge-connected cubic graph without an edge-3-coloring and with girth (the length of the shortest cycle) at least 5. Note that a graph is *cyclically k-edge-connected* if deleting fewer than k edges does not result in a graph having at least two components containing cycles. By [11], G must be cyclically 6-edge-connected and by Celmins [1], G has no circuit of length ≤ 6 (see also [8]).

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Suppose that H is a graph with exactly n (≥ 2) vertices of degree 2 and all other vertices of degree 3. We show that if a matrix M_H, characterized by H, has the same rank as another specified matrix M_n, then H cannot be a subgraph of G. In particular, using computer we prove that if H is a circuit of order $2 \leq n \leq 8$, then M_H and M_n have the same rank. Thus G has girth at least 9. Note that it is interesting to find the lower bounds for the girth of G, because we do not know whether there exists a cyclically 6-edge-connected snark with girth more than 6 (see [2, 4–6, 9, 10]).

2. Preliminaries

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. By a *multi-terminal network*, briefly a *network*, we mean a pair (G,U) where G is a graph and $U = (u_1,\dots,u_n)$ is an ordered set of pairwise distinct vertices of G. The vertices $u_1,\dots,u_n$ are called the *outer* vertices of (G,U) and the others are called the *inner* vertices of (G,U).

To each edge connecting u and v (including loops) we associate two distinct (directed) arcs, one directed from u to v, the other directed from v to u. If one of these arcs is denoted x then the other is denoted x^{-1}. Let $D(G)$ denote the set of such arcs, so that $|D(G)| = 2|E(G)|$. If $v \in V(G)$, then $\omega_G(v)$ denotes the set of arcs of G directed from v to $V(G) \setminus \{v\}$.

If G is a graph and A is an additive Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that $\partial\varphi(v) = \sum_{x\in\omega_G(v)} \varphi(x)$ ($v \in V(G)$) is called the *boundary* of φ. An A-chain φ in G is called *nowhere-zero* if $\varphi(x) \neq 0$ for every $x \in D(G)$. If (G,U) is a network, then an A-chain φ in G is called an *A-flow* in (G,U) if $\partial\varphi(v) = 0$ for every inner vertex v of (G,U).

By a (*nowhere-zero*) *A-flow* in a graph G we mean a (nowhere-zero) A-flow in the network $(G,\emptyset)$. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in Jaeger [3]. By Tutte [13, 14], a graph has a nowhere-zero k-flow if and only if it has a nowhere-zero A-flow for any Abelian group A of order k. Thus the study of nowhere-zero 5-flows is, in a certain sense, equivalent to the study of nowhere-zero $\mathbb{Z}_5$-flows. We use this fact and deal only with $\mathbb{Z}_5$-flows because they are easier to handle than integral flows.

A network (G,U), $U = (u_1,\dots,u_n)$, is called *simple* if the vertices $u_1,\dots,u_n$ have degree 1. If φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G,U), then denote by $\partial\varphi(U)$ the n-tuple $(\partial\varphi(u_1),\dots,\partial\varphi(u_n))$. By simple counting, $\sum_{i=1}^{n}\partial\varphi(u_i) = -\sum_{v\in V(G)\setminus U}\partial\varphi(v) = 0$ (see [7,9]). Furthermore, $\partial\varphi(u_i)\neq 0$ because u_i has degree 1 ($i=1,\dots,n$). Thus $\partial\varphi(U)$ belongs to the set

$$S_n = \{(s_1,\dots,s_n);\ s_1,\dots,s_n \in \mathbb{Z}_5 - \{0\},\ s_1+\cdots+s_n = 0\}.$$

For every $s\in S_n$, denote by $F_{G,U}(s)$ the number of nowhere-zero $\mathbb{Z}_5$-flows φ in (G,U) satisfying $\partial\varphi(U)=s$.

A partition $P=\{Q_1,\dots,Q_r\}$ of the set $\{1,\dots,n\}$, $n\geq 2$, is called *proper* if each of $Q_1,\dots,Q_r$ has cardinality at least 2. Let $\mathcal{P}_n$ denote the set of proper partitions of $\{1,\dots,n\}$ and let $p_n = |\mathcal{P}_n|$. If $s=(s_1,\dots,s_n)\in S_n$, $P=\{Q_1,\dots,Q_r\}\in\mathcal{P}_n$, and $\sum_{i\in Q_j} s_i = 0$ for $j=1,\dots,r$, then we say that P and s are *compatible*. (For example, $\{\{1,2\},\{3,4,5\}\}\in\mathcal{P}_5$ is compatible with $(1,4,1,2,2)\in S_5$.) In this paper, we consider $\mathcal{P}_n$ as an p_n-tuple $(P_{n,1},\dots,P_{n,p_n})$. For any $s\in S_n$, denote by $\chi_n(s)$ the integral vector $(c_{s,1},\dots,c_{s,p_n})$ so that $c_{s,i}=1$ ($c_{s,i}=0$) if $P_{n,i}$ is (is not) compatible with s, $i=1,\dots,p_n$.

Let (G,U), $U=(u_1,\dots,u_n)$, be a simple network and $\mathcal{H}$ be the set of minors of G which can be obtained by either contracting or deleting each edge of G which is not adjacent to a vertex in U. Thus each graph $H\in\mathcal{H}$ naturally corresponds to a partition $\pi(H)$ of U. By repeatedly applying Tutte's deletion/contraction formula [13,14] for enumerating nowhere-zero $\mathbb{Z}_5$-flows, one sees that the function $F_{G,U}(s)$ is a linear combination of the functions in $\{F_{H,U}(s): H\in\mathcal{H}, \pi(H)\in\mathcal{P}_n\}$. We may restate this observation as follows (see [11] for a detailed presentation).

Lemma 1. *Let (G,U), $U=(u_1,\dots,u_n)$, be a simple network. Then there exist integers $x_1,\dots,x_{p_n}$ such that for every $s\in S_n$, $F_{G,U}(s)=\sum_{i=1}^{p_n} c_{s,i}x_i$ where $(c_{s,1},\dots,c_{s,p_n})=\chi_n(s)$.*

3. The main result

Let V_n be the linear hull of $\{\chi_n(s); s\in S_n\}$ in $\mathbb{R}^{p_n}$.

Let H be a graph having exactly n (≥ 2) vertices of degree 2, namely $v_1,\dots,v_n$, and all other vertices of degree 3. Add to H vertices $u_1,\dots,u_n$ and edges $v_1u_1,\dots,v_nu_n$. Denote the resulting graph by H'. Consider the simple network (H',U), $U=(u_1,\dots,u_n)$. Let $S_H=\{s\in S_n; F_{H',U}(s)>0\}$ and V_H be the linear hull of $\{\chi_n(s); s\in S_H\}$ in $\mathbb{R}^{p_n}$. Then $V_H\subseteq V_n$.

Lemma 2. *If $V_H=V_n$, then H cannot be a subgraph of a smallest counterexample to the 5-flow conjecture.*

Proof. Let G be a counterexample to the 5-flow conjecture of the smallest possible order. As pointed out before, G is a 3-connected cubic graph. Suppose that H is a subgraph of G having minimum degree 2. Let $v_1,\dots,v_n$, $u_1,\dots,u_n$ and H' be as above (notice that all these vertices are pairwise different and $n\geq 2$).

Subdivide each edge from $E(G)\backslash E(H)$ having both ends among $v_1,\dots,v_n$ by a new vertex of degree 2 and denote the resulting graph by G'. Let $e_1,\dots,e_n$ be the edges from $E(G')\backslash E(H)$ incident with $v_1,\dots,v_n$, respectively. These edges are pairwise different. Take $I=G'-V(H)$. Let $v'_1,\dots,v'_n$ be the ends of $e_1,\dots,e_n$, respectively, which belong to $V(I)$ (notice that $v'_1,\dots,v'_n$ do not need to be pairwise different). Let I' be the graph arising from I after adding (pairwise different) vertices $w_1,\dots,w_n$ and edges $w_1v'_1,\dots,w_nv'_n$. Consider the simple network (I',W), $W=(w_1,\dots,w_n)$.

If there exists $s\in S_n$ such that $F_{H',U}(s)$, $F_{I',W}(s)>0$, then (H',U) and (I',W) have nowhere-zero $\mathbb{Z}_5$-flows φ_1 and φ_2, respectively, such that $\partial\varphi_1(U)=\partial\varphi_2(W)=s$ and the flows φ_1 and $-\varphi_2$ can be "pieced together" into nowhere-zero $\mathbb{Z}_5$-flows in G' and G, a contradiction. Thus $F_{H',U}(s)F_{I',W}(s)=0$ for every $s\in S_n$. Since $S_H=\{s\in S_n; F_{H',U}(s)>0\}$, we have $F_{I',W}(s)=0$ for every $s\in S_H$.

By Lemma 1, there exist integers $x_1,\dots,x_{p_n}$ such that for every $s\in S_n$, $F_{I',W}(s)=\sum_{i=1}^{p_n}c_{s,i}x_i$ where $(c_{s,1},\dots,c_{s,p_n})=\chi_n(s)$. Choose n-tuples $t_1,\dots,t_r$ from S_H so that $\chi_n(t_1),\dots,\chi_n(t_r)$ form a basis in $V_H=V_n$. Then for every $s\in S_n$, there are numbers $y_{s,1},\dots,y_{s,r}$ such that $\chi_n(s)=\sum_{j=1}^{r}y_{s,j}\chi_n(t_j)$ and, therefore, $F_{I',W}(s)=\sum_{i=1}^{p_n}c_{s,i}x_i=\sum_{i=1}^{p_n}(\sum_{j=1}^{r}y_{s,j}c_{t_j,i})x_i=\sum_{j=1}^{r}y_{s,j}(\sum_{i=1}^{p_n}c_{t_j,i}x_i)=\sum_{j=1}^{r}y_{s,j}F_{I',W}(t_j)=0$ (because $t_1,\dots,t_r\in S_H$). Thus $F_{I',W}(s)=0$ for every $s\in S_n$.

Choose an edge e of H and consider the cubic graph G'' homeomorphic with $G-e$. G'' is bridgeless (because G is 3-connected) and has a nowhere-zero $\mathbb{Z}_5$-flow (because G is a smallest counterexample to the 5-flow conjecture). This implies that $F_{I',W}(s)>0$ for some $s\in S_n$, a contradiction. Thus H cannot be a subgraph of G. ∎

We give a recursive formula for computing p_n. If $\{Q_1,\dots,Q_r\}\in\mathcal{P}_n$ and $n\in Q_j$, then $|Q_j|\in\{2,\dots,n-2,n\}$. For $i=2,\dots,n-2$ (resp. $i=n$), $\mathcal{P}_n$ contains exactly $\binom{n-1}{i-1}p_{n-i}$ (resp. 1) partitions such that the element n is contained in an i-element subset. Hence for any $n\geq 2$, we have (see also [12])

$$p_n=1+\sum_{i=2}^{n-2}\binom{n-1}{i-1}p_{n-i}\,. \tag{1}$$

Let $\mathcal{A}$ denote the automorphism group of $\mathbb{Z}_5$. The elements of $\mathcal{A}$ are $\alpha_0=\mathrm{id}$, $\alpha_1=(1,2,4,3)$, $\alpha_2=(1,4)(2,3)$ and $\alpha_3=(1,3,4,2)$. If $s=(s_1,\dots,s_n)\in S_n$

and $\alpha \in \mathcal{A}$, then denote $\alpha(s) = (\alpha(s_1), \ldots, \alpha(s_n)) \in S_n$. We say that s and $\alpha(s)$ are σ_n-equivalent. Clearly, $\chi_n(s) = \chi_n(\alpha(s))$ and $F_{U,G}(s) = F_{U,G}(\alpha(s))$ for any simple network (G,U) with n outer vertices (because if φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G,U), then so is $\alpha(\varphi)$).

Let $a_n = |S_n|/4$ be the number of σ_n-classes and $t_1, \ldots, t_{a_n}$ be pairwise non σ_n-equivalent elements of S_n. Let M_n be the $a_n \times p_n$ matrix so that the i-th row of M_n is $\chi_n(t_i)$, $i = 1, \ldots, a_n$. Note that the rowspace V_n of M_n does not depend on the choice of $t_1, \ldots, t_{a_n}$.

Let $a_H = |S_H|/4$. Without loss of generality we can suppose that $t_1, \ldots, t_{a_H}$ are elements of S_H. Thus S_H contains all $s' \in S_n$ which are σ_n-equivalent to $s \in \{t_1, \ldots, t_{a_H}\}$. Let M_H be the $a_H \times p_n$ submatrix of M_n so that the i-th row of M_H is $\chi_n(t_i)$, $i = 1, \ldots, a_H$. Note that the rowspace V_H of M_H does not depend on the choice of $t_1, \ldots, t_{a_H}$. Clearly, $V_H = V_n$ if and only if M_H and M_n have the same rank. Thus the following statement is a more convenient reformulation of Lemma 2 in matrix terms.

Lemma 3. *If the matrices M_H and M_n have the same rank, then H cannot be a subgraph of a smallest counterexample to the 5-flow conjecture.*

Clearly $a_1 = 0$ and $a_2 = 1$. Furthermore, S_n has exactly $4|S_{n-2}|$ ($3|S_{n-1}|$) n-tuples so that the sum of the last two elements is zero (nonzero). Thus $|S_n| = 4|S_{n-2}| + 3|S_{n-1}|$, whence $a_n = 4a_{n-2} + 3a_{n-1}$ for any $n \geq 3$. Therefore $a_5 = 51$, $a_6 = 205$, $a_7 = 819$, $a_8 = 3277$. By (1), we get that $p_5 = 11$, $p_6 = 41$, $p_7 = 162$, $p_8 = 715$.

Let C_n be the circuit (connected 2-regular graph) of order $n \geq 2$. It is easy to check that $S_{C_n} = S_n$, whence $V_{C_n} = V_n$ for any $n \leq 4$ (see, e.g., [7]). Using computer we can check that $a_{C_5} = 45$, $a_{C_6} = 151$, $a_{C_7} = 483$, $a_{C_8} = 1513$ and that the rank of M_{C_n} equals the rank of M_n for $n = 5, \ldots, 8$. More precisely, the rank of M_n is 11, 40, 147, 568 if $n = 5, 6, 7, 8$, respectively. Thus, by Lemma 3, the following statement holds true.

Theorem 4. *The smallest counterexample of the 5-flow conjecture has girth at least 9.*

In order to compute the ranks of M_{C_n} and M_n, we first create p_n permutations from $\mathcal{P}_n$, then a_n non σ_n-equivalent elements from S_n such that the first a_{C_n} elements are from S_{C_n}, and finally we can create rows of M_n. Now it suffices to modify the Gaussian elimination method so that we subsequently create rows of M_n during the computation and transform M_n to upper triangular form. This modification helps to save the space, because in any stage, we need the present row of M_n together with previous rows forming a maximal independent set. In such a way we check whether all rows of

M_n linearly depend on the rows from M_{C_n}. We have computed this problem on different personal computers using Maple 5 and Maple 6 programming languages. Note that the program runs several minutes for $n=7$, but several hours for $n=8$ (M_7 and M_8 are matrices of orders 819×162 and 3277×715, respectively).

One would expect that using very strong computers we can apply similar program also for $n \geq 9$. But it would be more interesting to develop more sophisticated methods, which do not use numerical computation, and can be applied for every $n \geq 9$. Really, if we give an affirmative answer to the following question, then we get a solution of the 5-flow conjecture.

Problem 5. Is $V_{C_n} = V_n$ for every $n \geq 2$?

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